# Regularity properties of LSD for certain families of random patterned matrices 

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(1) Introduction
(2) Wigner Matrix
(3) Discrete Schrödinger operator

4 Absolute continuity of Limiting Spectral Distribution
(5) Random (symmetric) Toeplitz/Hankel matrix
(6) Random Toeplitz matrix
(7) Random Hankel matrix
(8) Thank you

## Introduction

## Random matrix theory

Random matrix was first introduced by Wishart in mathematical statistics. It got significant attention after Wigner (in 1955) used the statistical properties of the eigenvalues of certain random matrix to study the nuclear energy levels.

## Anderson tight binding model

Another important model was proposed by P W Anderson (in 1958), for studying spin wave diffusion over doped semi-conductors. The model is described by a random self-adjoint operator on infinite dimensional Hilbert space.

## Introduction

The main question I will be focusing on here is about the absolute continuity of Limiting Spectral Distribution for random Toeplitz and Hankel matrices.

## Wigner Random Matrix

- Wigner matrix is

$$
M_{N}=\frac{1}{\sqrt{N}}\left(\begin{array}{cccc}
X_{1,1} & X_{1,2} & \cdots & X_{1, N} \\
X_{2,1} & X_{2,2} & \cdots & X_{2, N} \\
\vdots & \vdots & \ddots & \vdots \\
X_{N, 1} & X_{N, 2} & \cdots & X_{N, N}
\end{array}\right)
$$

where $X_{i, j}=X_{j, i}$ are random variables following $N(0,1)$.

- Let $\left\{E_{N, i}\right\}_{i}$ denote the eigenvalues of $M_{N}$ and define the empirical distribution of the eigenvalues by

$$
L_{N}(\cdot)=\frac{1}{N} \sum_{i} \delta_{E_{N, i}}(\cdot)
$$

- This is a random measure and one of the primary result is

$$
L_{N} \rightarrow \frac{1}{2 \pi} \sqrt{4-x^{2}} \chi_{\{|x|<2\}} d x
$$

weakly, almost surely.

## Wigner Random Matrix

- The limiting spectral distribution for the Wigner matrix is determined by computing the limits of moments and showing

$$
\lim _{N \rightarrow \infty} \frac{1}{N} \operatorname{tr}\left(M_{N}^{2 k}\right)=\frac{(2 k)!}{k!(k+1)!}
$$

Thus the limiting measure turns out to be the semi-circle law.

- This is clearly an absolutely continuous measure.


## Discrete Schrödinger operator

- On $\ell^{2}(\mathbb{Z})$ consider the operator

$$
\left(H^{\omega} u\right)_{n}=u_{n+1}+u_{n-1}+\omega_{n} u_{n} \quad n \in \mathbb{Z}, u \in \ell^{2}(\mathbb{Z})
$$

Where $\left\{\omega_{n}\right\}_{n}$ are i.i.d real random variable following distribution $\rho$.

- Consider the finite cut-off operator $H_{L}^{\omega}$ on $\ell^{2}(\{-N, \cdots, N\})$ and let $\left\{E_{L, i}\right\}_{i}$ to be its eigenvalues. Define

$$
L_{N}(\cdot)=\frac{1}{2 N+1} \sum_{i} \delta_{E_{L, i}}(\cdot)
$$

- Denoting $E_{H^{\omega}}$ to be the spectral measure for $H^{\omega}$, we know

$$
L_{N}(\cdot) \rightarrow \underset{\omega}{\mathbb{E}}\left[\left\langle\delta_{0}\right| E_{H^{\omega}}(\cdot)\left|\delta_{0}\right\rangle\right] \quad \text { a.s. }
$$

- In general, the limiting distribution may not be absolutely continuous.


## Absolute continuity of Limiting Spectral Distribution

## Borel-stieltjes transform

- One of the ways to study absolute continuity of a measure is through Borel-stieltjes transform.
- Given a probability measure $\mu$, the Borel-stieltjes transform is

$$
\hat{\mu}(z)=\int \frac{d \mu(t)}{t-z}
$$

- If $\mu$ is absolutely continuous, we can recover the density of $\mu$ using

$$
\frac{d \mu}{d x}(x)=\lim _{\epsilon \downarrow 0} \frac{1}{\pi} \Im \hat{\mu}(x+\iota \epsilon) \quad \text { a.e } x .
$$

## Absolute continuity of LSD for discrete Anderson model

- Therefore for regularity properties of the LSD, we study

$$
G(z)=\underset{\omega}{\mathbb{E}}\left[\Im\left\langle\delta_{0}\right|\left(H^{\omega}-z\right)^{-1}\left|\delta_{0}\right\rangle\right] \quad z \in \mathbb{C}^{+}
$$

- Using resolvent equation, one has

$$
\left\langle\delta_{0}\right|\left(H^{\omega}-z\right)^{-1}\left|\delta_{0}\right\rangle=\frac{1}{\omega_{0}-h^{\omega \mid \omega_{0}}(z)}
$$

where $h^{\omega \mid \omega_{0}}$ is independent of $\omega_{0}$ and satisfies

$$
\Im h^{\omega \mid \omega_{0}}(z)>0 \text { for } \Im z>0
$$

- Hence

$$
G(z)=\underset{\omega \mid \omega_{0}}{\mathbb{E}}\left[\Im \int \frac{\rho\left(\omega_{0}\right) d \omega_{0}}{\omega_{0}-h^{\omega \mid \omega_{0}}(z)}\right] \leq \pi\|\rho\|_{\infty}
$$

for $z \in \mathbb{C}^{+}$. In particular, the Limiting spectral distribution for discrete Schrödinger operator is absolutely continuous with $L^{\infty}$ density.

## Random (symmetric) Toeplitz/Hankel matrix

## Random (symmetric) Toeplitz/Hankel matrix

Let $\left\{X_{n}\right\}_{n=0}^{\infty}$ be sequence of i.i.d real random variable following $N(0,1)$ (normal distribution):

- Random Toeplitz matrix $M_{N}$ is given by

$$
\left[M_{N}\right]_{i, j}=\frac{1}{\sqrt{N}} X_{|i-j|} \quad 1 \leq i, j \leq N
$$

- Random Hankel matrix $M_{N}$ is given by

$$
\left[M_{N}\right]_{i, j}=\frac{1}{\sqrt{N}} X_{i+j-2} \quad 1 \leq i, j \leq N
$$

## Main result

## Theorem

Let $\left\{X_{n}\right\}_{n=0}^{\infty}$ be i.i.d real random variables following $N(0,1)$. Then

- The limiting spectral distribution for random Toeplitz matrix $\left[N^{-\frac{1}{2}} X_{|i-j|}\right]_{1 \leq i, j \leq N}$ is absolutely continuous with bounded density.
- The limiting spectral distribution for random Hankel matrix $\left[N^{-\frac{1}{2}} X_{i+j-2}\right]_{1 \leq i, j \leq N}$ is absolutely continuous with bounded density.


## Previous results

## Random (symmetric) Toeplitz matrix

- Bryc, Dembo and Jiang (2006) showed that the LSD exists for random (symmetric) Toeplitz and Hankel matrices. It was shown by the method of moments.
- Sen and Virag (2011) showed that the LSD for random Toeplitz matrix is absolutely continuous by embedding it in a Circulant matrix and using spectral averaging method.


## Random (symmetric) Toeplitz matrix

Given a random Toeplitz matrix $M_{N}=\left[N^{-\frac{1}{2}} X_{|i-j|}\right]_{1 \leq i, j \leq N}$ where $X_{n}$ are i.i.d r.v following $N(0,1)$ and $N$ is odd, we have

- We can write $M_{N}=A_{N}+B_{N}$.
- $A_{N}$ and $B_{N}$ are symmetric and independent of each other.
- We can write

$$
A_{N}=Y_{0}+\sum_{i=1}^{\frac{N-1}{2}} Y_{i}\left(C_{N}^{i}+\left(C_{N}^{*}\right)^{i}\right) \quad \& \quad B_{N}=\sum_{i=1}^{\frac{N-1}{2}} Z_{i}\left(D_{N}^{i}+\left(D_{N}^{*}\right)^{i}\right)
$$

where

$$
C_{N}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 \\
1 & 0 & \cdots & \cdots & 0
\end{array}\right) \quad \& D_{N}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 \\
-1 & 0 & \cdots & \cdots & 0
\end{array}\right)
$$

Set $Y_{i}=\frac{X_{i}+X_{N-i}}{2}$ with $Y_{0}=X_{0}$ and define $A_{N}$ to be

$$
\frac{1}{\sqrt{N}}\left(\begin{array}{cccccccc}
Y_{0} & Y_{1} & \cdots & Y_{\frac{N-1}{2}} & Y_{\frac{N-1}{2}} & \cdots & Y_{2} & Y_{1} \\
Y_{1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & Y_{2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
Y_{\frac{N-1}{2}} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & Y_{\frac{N-1}{2}} \\
Y_{\frac{N-1}{2}} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & Y_{\frac{N-1}{2}} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
Y_{2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & Y_{1} \\
Y_{1} & Y_{2} & \cdots & Y_{\frac{N-1}{2}} & Y_{\frac{N-1}{2}} & \cdots & Y_{1} & Y_{0}
\end{array}\right)
$$

Set $Z_{i}=\frac{X_{i}-X_{N-i}}{2}$ and define $B_{N}$ to be
$\frac{1}{\sqrt{N}}\left(\begin{array}{cccccccc}0 & Z_{1} & \cdots & Z_{\frac{N-1}{2}} & -Z_{\frac{N-1}{2}} & \cdots & -Z_{2} & -Z_{1} \\ Z_{1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -Z_{2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ Z_{\frac{N-1}{2}} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -Z_{\frac{N-1}{2}} \\ -Z_{\frac{N-1}{2}} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & Z_{\frac{N-1}{2}} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -Z_{2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & Z_{1} \\ -Z_{1} & -Z_{2} & \cdots & -Z_{\frac{N-1}{2}} & Z_{\frac{N-1}{2}} & \cdots & Z_{1} & 0\end{array}\right)$

## Random (symmetric) Toeplitz matrix

- The eigenvectors of $A_{N}$ and $B_{N}$ are fixed.
- The eigenvalues are given by

$$
\begin{aligned}
& E_{A_{N}, j}=\frac{Y_{0}}{\sqrt{N}}+\frac{2}{\sqrt{N}} \sum_{i=1}^{\frac{N-1}{2}} Y_{i} \cos \frac{2 \pi i j}{N} \\
& E_{B_{N}, j}=\frac{2}{\sqrt{N}} \sum_{i=1}^{\frac{N-1}{2}} Z_{i} \cos \frac{\pi(2 j+1) i}{N}
\end{aligned}
$$

- The eigenvalues follows gaussian distribution and

$$
\mathbb{E}\left[E_{A_{N}, i} E_{A_{N}, j}\right]=\frac{1}{2} \delta_{i-j}-\frac{1}{2 N}+\frac{1}{2} \delta_{i} \delta_{j} \quad 0 \leq i, j \leq \frac{N-1}{2} .
$$

## Random (symmetric) Toeplitz matrix

- Since eigenvectors are independent of the configuration, we can write

$$
A_{N}=\sum_{i=0}^{\frac{N-1}{2}} E_{A_{N}, i} P_{i}
$$

where $P_{i}$ are rank 2 projections.

- Viewing

$$
M_{N}=A_{N}+B_{N}=B_{N}+\sum_{i} E_{A_{N}, i} P_{i}
$$

where $E_{A_{N}, i}$ are independent real gaussian random variables.

- Defining

$$
\tilde{M}_{N, i}=B_{N}+\sum_{j \neq i} E_{A_{N}, j} P_{j}
$$

we can view $E_{A_{N}, i}$ as independent of $\tilde{M}_{N, i}$.

## Random (symmetric) Toeplitz matrix

- To study the LSD, we are going to use the Borel-stieltjes transform, hence

$$
\frac{1}{N} \mathbb{E}\left[\Im \operatorname{tr}\left(\left(M_{N}-z\right)^{-1}\right)\right] \quad z \in \mathbb{C}^{+}
$$

- We have

$$
\begin{aligned}
& \mathbb{E}\left[\Im \operatorname{tr}\left(\left(M_{N}-z\right)^{-1}\right)\right] \\
& \quad=\sum_{i=0}^{\frac{N-1}{2}} \mathbb{E}\left[\Im \operatorname{tr}\left(P_{i}\left(\tilde{M}_{N, i}+E_{A_{N}, i} P_{i}-z\right)^{-1} P_{i}\right)\right]
\end{aligned}
$$

where we can use the spectral averaging technique.

## Spectral Averaging result

- Given a self adjoint operator $A$ and a normalized vector $\psi$ in a separable Hilbert space $\mathscr{H}$ set $A_{\lambda}=A+\lambda|\psi\rangle\langle\psi|$. Then

$$
\int\left\langle\psi, E_{A_{\lambda}}(\cdot) \psi\right\rangle d \lambda=D|\cdot|
$$

where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}$.

- The perturbation can be replaced by a bounded non-negative operator with an appropriate modification of the integrand.


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- The perturbation can be replaced by a bounded non-negative operator with an appropriate modification of the integrand.


## Spectral averaging for rank 2 perturbation

Given a self adjoint operator $A$ and a rank 2 projection $P$, define $A_{\lambda}=A+\lambda P$, then for a probability density $\rho$ with bounded density

$$
\int \operatorname{tr}\left(P E_{A_{\lambda}}(I) P\right) \rho(\lambda) d \lambda \leq D\|\rho\|_{\infty}|I| .
$$

## Random Hankel matrix

Similar to Toeplitz matrix case, given random Hankel matrix $M_{N}=\left[N^{-\frac{1}{2}} X_{i+j-2}\right]_{1 \leq i, j \leq N}$, where $X_{n}$ are i.i.d r.v following $N(0,1)$ and $N$ is odd, we have

- Write $M_{N}=\tilde{A}_{N}+\tilde{B}_{N}$
- $\tilde{A}_{N}$ and $\tilde{B}_{N}$ are symmetric and independent of each other.
- Let $\phi_{\theta}=\left(1, e^{\iota \theta}, \cdots, e^{\iota(N-1) \theta}\right)^{t}$. Then we have the relations

$$
\tilde{A}_{N} \phi_{\frac{2 \pi j}{N}}=\lambda_{j} \phi_{-\frac{2 \pi j}{N}}
$$

where

$$
\lambda_{j}=\frac{1}{\sqrt{N}} \sum_{i} Y_{i} e^{\ell \frac{2 \pi j(i-1)}{N}},
$$

which can be compactly written as

$$
\tilde{A}_{N}\binom{\phi_{\frac{2 \pi j}{}}^{N}}{\phi_{-\frac{2 \pi j}{N}}^{N}}=\left(\begin{array}{cc}
0 & \lambda_{j} \\
\lambda_{j} & 0
\end{array}\right)\binom{\phi_{\frac{2 \pi j}{}}^{N}}{\phi_{-\frac{2 \pi j}{N}}} .
$$

$$
\tilde{A}_{N}=\frac{1}{\sqrt{N}}\left(\begin{array}{ccccc}
Y_{1} & Y_{2} & \cdots & Y_{N-1} & Y_{N} \\
Y_{2} & Y_{3} & \cdots & Y_{N} & Y_{1} \\
\vdots & \vdots & . & . & \vdots \\
Y_{N-1} & Y_{N} & . & . & \vdots \\
Y_{N} & Y_{1} & \cdots & Y_{N-2} & Y_{N-1}
\end{array}\right)
$$

here $Y_{i}=\frac{X_{i}+X_{N+i}}{2}\left(\right.$ will use $\left.Y_{N}=X_{N}\right)$

$$
\tilde{B}_{N}=\frac{1}{\sqrt{N}}\left(\begin{array}{ccccc}
Z_{1} & Z_{2} & \cdots & Z_{N-1} & 0 \\
Z_{2} & Z_{3} & \cdots & 0 & -Z_{1} \\
\vdots & \vdots & . & . & \vdots \\
Z_{N-1} & 0 & . & . & -Z_{N-2} \\
0 & -Z_{1} & \cdots & -Z_{N-2} & -Z_{N-1}
\end{array}\right) \text {, }
$$

here $Z_{i}=\frac{X_{i}-X_{N+i}}{2}$.

## Random Hankel matrix

- Denoting $L_{j, \lambda}=\lambda\left|\phi_{\frac{2 \pi j}{N}}\right\rangle\left\langle\phi_{-\frac{2 \pi j}{N}}\right|+\bar{\lambda}\left|\phi_{-\frac{2 \pi j}{N}}\right\rangle\left\langle\phi_{\frac{2 \pi j}{N}}\right|$ and

$$
\begin{aligned}
P_{j} & =\left|\phi_{\frac{2 \pi j}{N}}\right\rangle\left\langle\phi_{\frac{2 \pi j}{N}}\right|+\left|\phi_{-\frac{2 \pi j}{N}}\right\rangle\left\langle\phi_{-\frac{2 \pi j}{N}}\right| \text { we can write } \\
& \mathbb{E}\left[\operatorname{tr}\left(\left(M_{N}-z\right)^{-1}\right)\right]=\sum_{j=1}^{\frac{N-1}{2}} \mathbb{E}\left[\operatorname{tr}\left(P_{j}\left(\tilde{M}_{N, j}+L_{j, \lambda_{j}}-z\right)^{-1} P_{j}\right)\right]
\end{aligned}
$$

we can follow similar steps as before.

- For absolute continuity, we need to estimate

$$
\lim _{\Im z \downarrow 0} \int_{\mathbb{C}} \Im \operatorname{tr}\left(P_{j}\left(\tilde{M}_{N, j}+L_{j, \lambda}-z\right)^{-1} P_{j}\right) e^{-|\lambda|^{2}} d \lambda
$$

- The main problem is $L_{j, \lambda}$ is not a non-negative operator and so spectral averaging does not work.


## Random Hankel matrix

- Note that $M_{N}$ is real symmetric, so all the eigenvectors are in $\mathbb{R}^{N}$.
- Also note that $\left\langle\phi_{\theta}, \psi\right\rangle=\overline{\left\langle\phi_{-\theta}, \psi\right\rangle}$ for $\psi \in \mathbb{R}^{N}$.
- Denoting $\left\{E_{n}\right\}_{n}$ to be eigenvalues of $M_{N}$ and $\left\{\psi_{n}\right\}_{n}$ as the eigenfunctions, we have

$$
\left\langle\phi_{\theta}\right|\left(M_{N}-z\right)^{-1}\left|\phi_{\theta}\right\rangle=\sum_{n} \frac{\left|\left\langle\phi_{\theta}, \psi\right\rangle\right|^{2}}{E_{n}-z}=\left\langle\phi_{-\theta}\right|\left(M_{N}-z\right)^{-1}\left|\phi_{-\theta}\right\rangle
$$

- All these leads to estimating integrals of the form

$$
\lim _{\epsilon \downarrow 0} \int_{\mathbb{C}} \frac{\epsilon f(\lambda)}{g(\lambda)^{2}+\epsilon^{2} h(\lambda)^{2}} d \lambda
$$

for certain $f, g, h: \mathbb{C} \rightarrow \mathbb{R}$ which are smooth.

## Random Hankel matrix

- Using co-area formula, previous expression is given by

$$
\pi \int_{g^{-1}\{0\}} \frac{f(\lambda)}{|\nabla g(\lambda)| h(\lambda)} d \sigma(\lambda)
$$

where $\sigma$ is the line measure on $g^{-1}\{0\}$.

- In the case of random Hankel matrix, upper bounding above expression boils down to

$$
\sup _{\substack{z_{0} \in \mathbb{C} \\ r>0}} r \int_{0}^{2 \pi} e^{-\left|z_{0}+r e^{\iota \theta}\right|^{2}} d \theta
$$

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$$

This concludes the proof of absolute continuity.


