On Hydrodynamic Limits of Young Diagrams

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Outline

- Motivations
- ‘Static’ Limits and ’Dynamic’ Model
- Main Results
- Sketch of Proof
- Some other possible directions

Motivations

In 1996, Vershik showed an interesting limit theorem for 2D Young diagrams sampled via certain ensembles, in particular the uniform ensemble.

–Later, we will refer to this type of limit as a ‘static’ limit.

More recently, in 2010, Funaki and Sasada considered an associated stochastic evolution of these 2D diagrams in time, and showed a hydrodynamic limit.

–This is what we mean by a ‘dynamical’ limit.

In this talk, we consider the behaviors under different Young diagram ensembles and evolutions.

–These different models provide a richer setting, allowing for different scalings, and ‘abstract modeling’ say of ‘polymer dynamics’ for instance.

–In particular, the hydrodynamic limits found depend on the type of ensemble chosen. The method of proof differs from previous work.

Notation on 2D Young diagrams

Let $p = (p_1, p_2, p_3, \ldots, p_n)$, where $p_k \geq p_{k+1}$, be a partition of the integer

$$M(p) = \sum_{k=1}^{n} p_k.$$

For example, $p = (4, 2, 2, 1)$ is a partition of

$$9 = 4 + 2 + 2 + 1$$

Let

$$\xi(k) \;=\; \#\{m : p_m = k\}$$

be the count of ‘particles’ or ‘polymers’ of length k.

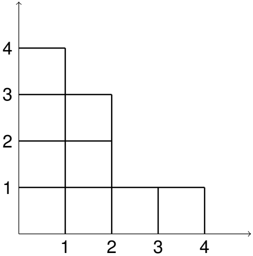
–So, $\xi = (1, 2, 0, 1, 0, \ldots)$

Define the ‘height’ function

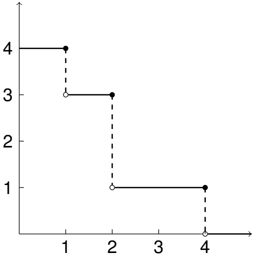
$$\psi(x) = \sum_{k \geq x} \xi(k).$$

Then, the associated ‘Young diagram’ is the graph of ψ.

The Young diagram corresponding to $p = (4, 2, 2, 1)$ or $\xi = (1, 2, 0, 1, 0 \ldots)$.



$p = (4, 2, 2, 1)$

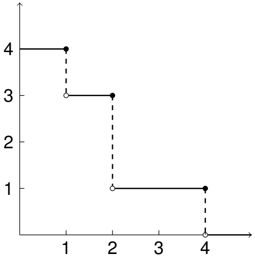


$\psi(x) = \psi(x; p)$

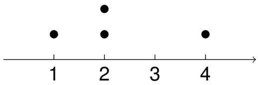
The number of squares equals

$$M(p) := \sum_{k=1}^{n} p_k = \sum_{k=1}^{n} k\xi(k) = \int_0^\infty \psi(x)dx.$$

Moreover, $\xi(k) = \psi(k) - \psi(k+1)$ can be viewed as negative gradient of ψ at k:



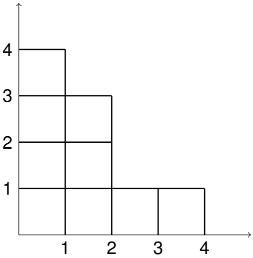
$\psi(x)$



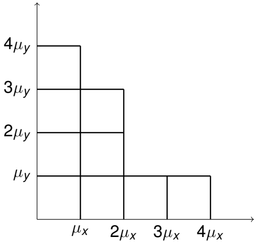
$\xi = (1, 2, 0, 1, 0, \ldots)$

To take limits as $M \to \infty$, we rescale the diagram widths and heights, say by μ_x and μ_y, to be chosen later.

After rescaling, the area of the Young diagram is $\mu_x \, \mu_y \, M$, which we would like to be $O(1)$.



Before scaling



After scaling

A result of Vershik '96

Let P_M be the uniform probability on all partitions of M,

e.g. $(4, 2, 2, 1)$ and $(5, 4)$ are equally likely with respect to $M = 9$.

For any partition p, consider the rescaled shape function

$$\psi_M(x; p) = \frac{1}{\sqrt{M}} \psi(x\sqrt{M}; p)$$

–So, here, $\mu_x = \mu_y = M^{-1/2}$.

Result

As $M \to \infty$, ψ_M concentrates near

$$\psi(x) = -\frac{\sqrt{6}}{\pi} \ln\left(1 - e^{-\pi x/\sqrt{6}}\right).$$

–That is,

$$\lim_{M\to\infty} P_M \left\{ \sup_{x\in[a,b]} |\psi_M(x;p) - \psi(x)| > \varepsilon \right\} = 0.$$

There are also other ‘static’ limits with respect to other ensembles, say, Haar statistics, and Plancherel, Ewens measures:

Erlihson-Granovsky ’08,
Kerov-Vershik ’77,
Yakubovich ’12, etc.

–We will describe the ones introduced in Fatkullin-Slastikov ’18

The uniform measure P_M can be thought as the canonical ensemble corresponding to the grand canonical ensemble

$$\mathcal{P}_\kappa(\xi) = \frac{1}{Z_\kappa} e^{-\kappa M(\xi)}.$$

–Here $\kappa > 0$ is a chemical potential.

Consider the family of grand canonical ensembles in Fatkullin-Slastikov '18, which includes $\mathcal{P}_\kappa$:

$$\mathcal{P}_{\beta,N}(\xi) = \frac{1}{Z_{\beta,N}} e^{-\beta \sum_{k\geq 1} \xi(k)\mathcal{E}_k - N^{-1}M(\xi)}.$$

Here $\beta \geq 0$ is the inverse temperature,

and we have taken $\kappa = N^{-1}$ as the chemical potential, in terms of a scaling parameter N.

Observe that the grand canonical ensemble

$$\mathcal{P}_{\beta,N}(\xi) = \frac{1}{Z_{\beta,N}} e^{-\beta \sum_{k\in\mathbb{N}} \xi(k)\mathcal{E}_k - N^{-1}\sum_{k\in\mathbb{N}} k\xi(k)}.$$

has a product structure:

$$\mathcal{P}_{\beta,N}(\xi) = \prod_{k=1}^{\infty} \mathcal{P}_{\beta,N,k}(\xi(k))$$

where $\mathcal{P}_{\beta,N,k}$ is Geometric with parameter

$$\theta_k = e^{-\beta\mathcal{E}_k - k/N},$$

that is, for $n \geq 0$,

$$\mathcal{P}_{\beta,N,k}(n) = (1-\theta_k)\theta_k^n.$$

Each size k carries in a sense an energy $\mathcal{E}_k$.

–When $\beta > 0$, if $\mathcal{E}_k >> \ln k$, the measure concentrates on finitely many particles.

So, we consider $\mathcal{E}_k$ in the form

$$\mathcal{E}_k = u(\ln k)$$

where $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+_\circ$ is a function with properties:

$u(\cdot)$ is differentiable, and $u'(\cdot)$ is bounded.

Also, $\lim_{x\to\infty} u(x) = \infty$, and $\lim_{x\to\infty} u'(x) = 0$ or 1.

–For instance,

- $u(x) \sim x$ or
- $u(x) \sim \ln(x)$ would be fine.

These are instances in the two classes that we discuss here:

- '$\mathcal{E}_k \sim \ln k$' denotes the case $\lim_{x\to\infty} u'(x) = 1$
- '$1 \ll \mathcal{E}_k \ll \ln k$' stands for the case $\lim_{x\to\infty} u'(x) = 0$.

–We note the measure, in the case '$1 \ll \mathcal{E}_k \ll \ln k$', penalizes less the number of particles with large size k, than in the case '$\mathcal{E}_k \sim \ln k$'.

‘Static’ limit results in FS ’18

Let $N_\beta = e^{\beta \mathcal{E}_k}$.

From our assumptions,

$$N_\beta = \begin{cases} 1 & \text{when } \beta = 0 \\ o(N) & \text{when } \beta > 0. \end{cases}$$

One can show that, with respect to $\mathcal{P}_{\beta,N}$, that there are an order

$$E[M] = N^2 N_\beta^{-1}$$

number of squares under the diagram.

–We will rescale $\mu_x = 1/N$ and $\mu_y = N_\beta/N$ and form

$$\psi_N(x) := \frac{N_\beta}{N}\psi(Nx).$$

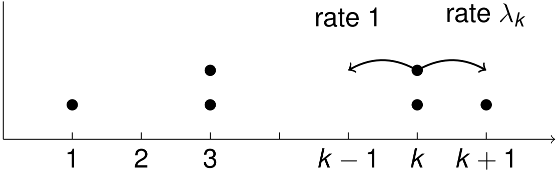
Then, as $N \to \infty$, in mean, it is shown in FS '18 that

- $\beta = 0$: $\psi_N(x) \to \dfrac{6}{\pi^2} \ln(1 - e^{-x})$
- $\mathcal{E}_k \sim \ln k$, $0 < \beta < 1$: $\psi_N(x) \to \dfrac{1}{\Gamma(2-\beta)} \displaystyle\int_x^\infty u^{-\beta} e^{-u} du$
- $1 \ll \mathcal{E}_k \ll \ln k$, $\beta > 0$: $\psi_N(x) \to e^{-x}$

–When $\beta \geq 1$ and '$\mathcal{E}_k \sim \ln k$', it turns out the variance of $\psi_N(x)$ diverges; we exclude this case here.

Dynamics

For our evolutional models, consider the following weakly asymmetric zero range processes on $\mathbb{Z}^+$.



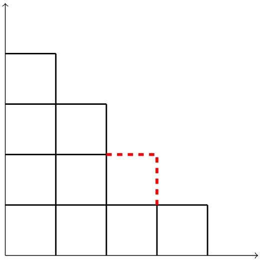
–We impose a reflecting boundary at $k = 1$, so that the gradient evolution is ‘mass conservative’.

Namely, we consider generator

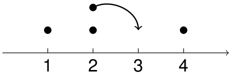
$$Lf(\xi) = \sum_{k=1}^{\infty} \Big\{ \lambda_k \left[f\left(\xi^{k,k+1}\right) - f(\xi) \right] \chi_{\{\xi(k)>0\}} + \left[f\left(\xi^{k,k-1}\right) - f(\xi) \right] \chi_{\{\xi(k)>0,k>1\}} \Big\}$$

where

$$\lambda_k = e^{-\beta(\mathcal{E}_{k+1}-\mathcal{E}_k)-N^{-1}}, \quad \xi^{x,y}(k) = \begin{cases} \xi(k)-1 & k = x \\ \xi(k)+1 & k = y \\ \xi(k) & \text{otherwise} \end{cases} .$$

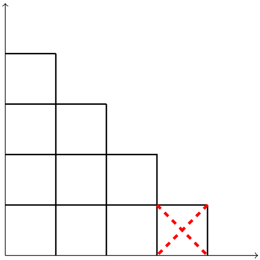


Growth at $(2, 1)$

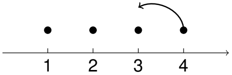


a particle jumps from site 2 to 3

In this example, a particle at site 2 jumps (with rate λ_2) to site 3 corresponds to creation of a square at the corner $(2, 1)$.



Loss at $(3, 0)$



a particle jumps from site 4 to 3

Here, a particle at site 4 jumps (with rate λ_4) to site 3 corresponds to annihilation of a square at the corner $(3, 0)$.

Invariant measures

Part of the reason for this choice of dynamics is that it keeps $\mathcal{P}_{\beta,N}$ invariant. But, there are other invariant measures.

Let

$$c_0 = \min_k e^{\beta \mathcal{E}_k}.$$

Trivially $c_0 = 1$ when $\beta = 0$ and $c_0 \geq 1$ otherwise.

For fixed β and $0 \leq c \leq c_0$, we introduce the product measures

$$\mathcal{R}_{c,N}(\xi) = \prod_k \mathcal{R}_{\beta,c,N,k}(\xi(k)).$$

Here, the marginal $\mathcal{R}_{\beta,c,N,k}$ is the Geometric distribution with parameter

$$\theta_{k,c} = c\theta_k = ce^{-\beta\mathcal{E}_k - k/N}.$$

Clearly, $\mathcal{R}_{c,N} = \mathcal{P}_{\beta,N}$ when $c = 1$.

–These measures can be seen to be invariant since

$$\lambda_k = e^{-\beta(\mathcal{E}_{k+1} - \mathcal{E}_k) - N^{-1}} = \frac{\theta_{k+1,c}}{\theta_{k,c}},$$

and so $\theta_{\cdot,c}$ is a reversible measure for the underlying Birth-Death chain.

Order of number of particles

Recall $N_\beta = e^{\mathcal{E}_N}$, which is $o(N)$ and diverges when $\beta > 0$.

When $c < c_0$,

$$E_{\mathcal{R}_{c,N}} \sum_{k=1}^{\infty} \xi(k) = O\left(\frac{N}{N_\beta}\right).$$

–So, there are $o(N)$ particles in the system under $\mathcal{R}_{c,N}$ when $\beta > 0$.

Static limits under $\mathcal{R}_{c,N}$

Proposition

Fix $0 \leq c \leq c_0$. Then, for any test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$ and $\delta > 0$

$$\lim_{N\to\infty} \mathcal{R}_{c,N} \left[\left| \frac{N_\beta}{N} \sum_{k=1}^{\infty} G(k/N)\xi(k) - \int_0^\infty G(x)\phi_c(x)dx \right| > \delta \right] = 0 \tag{1}$$

where ϕ_c takes the form

(1) $\phi_c = \dfrac{ce^{-x}}{1 - ce^{-x}}$ *when $\beta = 0$,*

(2) $\phi_c = cx^{-\beta}e^{-x}$ *when $\mathcal{E}_k \sim \ln k$ and $0 < \beta < 1$,*

(3) $\phi_c = ce^{-x}$ *when $1 \ll \mathcal{E}_k \ll \ln k$ and $\beta > 0$.*

–Limits for ψ_N in terms of $\int_x^\infty \phi_c(u)du$ also follow.

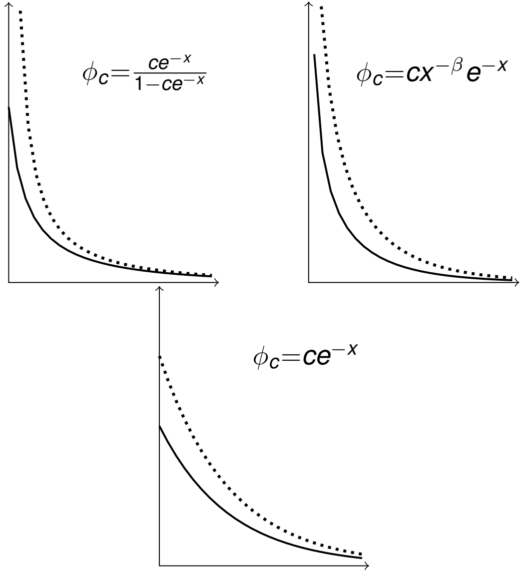


Figure: Examples of ϕ_c in all the three regimes. The dotted curves represent $c = c_0$ and solid curves are for general c's which are strictly less than c_0.

Rescaled empirical measure

Let ξ_t denote the associated Markov process generated by L. Since

$$\lambda_k = e^{-\beta(\mathcal{E}_{k+1}-\mathcal{E}_k)-N^{-1}} \to 1 \text{ as } N \to \infty,$$

we will be interested in the process

$$\eta_t = \xi_{N^2 t}$$

–For each t, consider the corresponding rescaled empirical measure

$$\pi_t^N(dx) = \frac{N_\beta}{N} \sum_{k=1}^{\infty} \eta_t(k)\delta_{k/N}(dx).$$

–We observe ξ_t in diffusive scale, where time is speeded up by N^2 and space by N. The extra factor N_β is needed to keep π_t^N nontrivial.

Initial measures

Consider a smooth initial density profile $\rho_0 : \mathbb{R}^+_\circ \to \mathbb{R}^+$ such that $\rho_0 \in L^1(\mathbb{R}^+)$.

Correspondingly, define a sequence of **'local equilibrium'** measures $\{\nu^N\}_{N\in\mathbb{N}}$ corresponding to ρ_0:

1. For all $N \in \mathbb{N}$ and $\eta \in \Omega$, $\nu^N(\eta) = \prod_{k=1} \nu_k^N(\eta(k))$ with ν_k^N Geometric distributions with parameter $\theta_{N,k}$.
2. $\lim_{N\to\infty} \dfrac{1}{N} \sum_{k=1}^{\infty} |N_\beta \rho_{N,k} - \rho_0(k/N)| = 0$ where $\rho_{N,k} := \dfrac{\theta_{N,k}}{1-\theta_{N,k}}$ is the mean of μ_k^N.
3. μ^N is stochastically bounded by $\mathcal{R}_{c,N}$ for some $0 \le c < c_0$.

–This last item means that $\rho_0 \le \phi_c$ for some $0 \le c < c_0$.

Theorem (Case $\beta = 0$)

Suppose $\beta = 0$ and $\rho_0 \in L^1(\mathbb{R}^+)$. Then, for any $t \geq 0$, test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_N\Big[\Big|\langle G, \pi_t^N\rangle - \int_0^\infty G(x)\rho(t,x)dx\Big| > \delta\Big] = 0,$$

where $\rho(t,x)$ is the unique weak solution in $\mathcal{C}$ of the equation

$$\begin{cases} \partial_t \rho = \partial_x^2 \dfrac{\rho}{\rho+1} + \partial_x \dfrac{\rho}{\rho+1} \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \displaystyle\int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \leq \phi_c(\cdot) \text{ for all } t \leq T \end{cases} .$$

Theorem (Case $\mathcal{E}_k \sim \ln k$)

Suppose $\mathcal{E}_k \sim \ln k$, $0 < \beta < 1$ and $\rho_0 \in L^1(\mathbb{R}^+)$. Then, for any $t \geq 0$, test function $G \in C_c^\infty(\mathbb{R}_\circ^+)$, and $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_N\left[\left|\langle G, \pi_t^N\rangle - \int_0^\infty G(x)\rho(t,x)dx\right| > \delta\right] = 0,$$

where $\rho(t,x)$ is the unique weak solution in $\mathcal{C}$ of the equation

$$\begin{cases} \partial_t \rho = \partial_x^2 \rho + \partial_x \left(\dfrac{\beta + x}{x}\rho\right) \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \displaystyle\int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \leq \phi_c(\cdot) \text{ for all } t \leq T \end{cases} . \tag{2}$$

Theorem (Case $1 \ll \mathcal{E}_k \ll \ln k$)

Suppose $1 \ll \mathcal{E}_k \ll \ln k$, $\beta > 0$ *and* $\rho_0 \in L^1(\mathbb{R}^+)$. *Then, for any* $t \geq 0$, *test function* $G \in C_c^\infty(\mathbb{R}_\circ^+)$, *and* $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_N\left[\left|\langle G, \pi_t^N\rangle - \int_0^\infty G(x)\rho(t,x)dx\right| > \delta\right] = 0,$$

where $\rho(t,x)$ *is the unique weak solution in* $\mathcal{C}$ *of the equation*

$$\begin{cases} \partial_t \rho = \partial_x^2 \rho + \partial_x \rho \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \leq \phi_c(\cdot) \text{ for all } t \leq T \end{cases} . \tag{3}$$

–Here, $\mathcal{C}$ is the space of functions $\rho : [0, T] \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that the map $t \in [0, T] \mapsto \rho(t, x)dx$ is vaguely continuous; that is, for each $G \in C_c^\infty(\mathbb{R}_\circ^+)$, the map

$$t \in [0, T] \mapsto \int_0^\infty G(x)\rho(t, x)dx$$

is continuous.

Height functions

Since the particle density is related to the height function by

$$\psi_N(t,x) = \frac{N_\beta}{N} \sum_{k \geq xN} \eta_t(k),$$

we obtain, as a corollary, that the macroscopic limit ψ satisfies

$$\psi(t,x) = \int_x^\infty \rho(t,u)du$$

and

- $\beta = 0$: $\partial_t \psi = \partial_x \left(\frac{\partial_x \psi}{1 - \partial_x \psi} \right) + \frac{\partial_x \psi}{1 - \partial_x \psi}$;
- $0 < \beta < 1$, $\mathcal{E}_k \sim \ln k$: $\partial_t \psi = \partial_x^2 \psi + \frac{\beta + x}{x} \partial_x \psi$;
- $\beta > 0$, $1 \ll \mathcal{E}_k \ll \ln k$: $\partial_t \psi = \partial_x^2 \psi + \partial_x \psi$.

We remark the third equation is a linearization of the first nonlinear PDE, while the middle linear PDE represents a 'critical' case in which the inhomogeneity of the rates appears.

–When $\beta > 0$, the expected number

$$\sup_{aN \leq k \leq bN} E_{\mathcal{R}_{c,N}}[\eta_t(k)] \sim N_\beta^{-1} \to 0$$

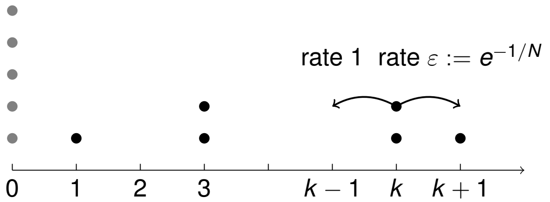
as $N \uparrow \infty$.

–When, however, $\beta = 0$, this expectation is $O(1)$.

Contrast with Funaki-Sasada model

Funaki and Sasada [FuSa] obtained the same equation, in the case $\beta = 0$, for a different model with a different initial condition.

- [FuSa] model: a weakly asymmetric reservoir at site 0



–Evolution is not conservative. The invariant measure is $\mathcal{P}_{0,N}$ which supports an infinite number of particles. Allowed initial densities ρ_0 satisfy $\int_0^\infty \rho_0(u)du = \infty$.

–In our limits, however, $\rho_0 \in L^1(\mathbb{R}^+)$.

Formal derivation of macroscopic equations

We have

$$\left\langle G, \pi_t^N \right\rangle = \left\langle G, \pi_0^N \right\rangle + \int_0^t N^2 L \left\langle G, \pi_s^N \right\rangle ds + M_t^G$$

with

$$\begin{aligned} N^2 L \left\langle G, \pi_t^N \right\rangle = {} & N^{-1} \sum_{k=2}^{\infty} \Delta_N G(k/N)\, N_\beta \chi_{\{\eta_t(k)>0\}} \\ & + N^{-1} \sum_{k=2}^{\infty} \frac{\lambda_k - 1}{1/N} \nabla_\mu G(k/N)\, N_\beta \chi_{\{\eta_t(k)>0\}} \end{aligned}$$

To capture the ‘drift’ term, recall

$$\lambda_k == e^{-\beta(\mathcal{E}_{k+1}-\mathcal{E}_k)-N^{-1}}.$$

Note, as $k/N \to x$ that

$$\frac{\lambda_k - 1}{1/N} \to \begin{cases} -1 & \beta = 0 \\ -\dfrac{\beta + x}{x} & \mathcal{E}_k \sim \ln k \\ -1 & 1 \ll \mathcal{E}_k \ll \ln k \end{cases}$$

–Let now

$$\eta^l(k) = \frac{1}{2l+1} \sum_{|j-k| \leq l} \eta(j).$$

The system relaxes such that locally it is in an 'equilibrium', that is in a window of length $N\epsilon$, roughly $o(1)$ macroscopically,

$$N_\beta \chi_{\eta(k)>0} \sim \frac{N_\beta \eta^{N\epsilon}(k)}{1 + \eta^{N\epsilon}(k)}.$$

Notice that typically $N_\beta \eta^{N\epsilon}(k) \sim \rho(t,x)$ then

$$\frac{N_\beta \eta^{N\epsilon}(k)}{1 + \eta^{N\epsilon}(k)} \sim \frac{\rho(t,x)}{1 + N_\beta^{-1} \rho(t,x)}$$

- $\beta = 0$: $N_\beta = 1$, so $N_\beta \chi_{\eta(k)} \sim \dfrac{\rho(t,x)}{1 + \rho(t,x)}$
- $\mathcal{E}_k \sim \ln k$ or $1 \ll \mathcal{E}_k \ll \ln k$: $N_\beta \to \infty$, so $N_\beta \chi_{\eta(k)} \sim \rho(t,x)$

Brief sketch of proof

To make the approximations rigorous, the general idea is from the 'entropy' method in Guo-Papanicolaou-Varadhan '89.

- 1-block estimate:

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}^N\left|\int_0^T N_\beta\left(\chi_{\eta_t(k)>0}-\frac{\eta_t^l(k)}{1+\eta_t^l(k)}\right)dt\right|=0.$$

- 2-block estimate when $\beta = 0$ and $N_\beta = 1$.

$$\limsup_{l\to\infty}\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\sup_{aN\le k\le bN}\mathbb{E}^N\left|\int_0^T\left(\frac{\eta_t^l(k)}{1+\eta_t^l(k)}-\frac{\eta_t^{\epsilon N}(k)}{1+\eta_t^{\epsilon N}(k)}\right)dt\right| = 0.$$

The 1-block estimate is sufficient for the cases when $\beta \neq 0$.

–One can expand

$$\frac{\eta^l(k)}{1+\eta^l(k)} \sim \eta^l(k) - \big(\eta^l(k)\big)^2 + \cdots$$

and then show for instance

$$\sup_{aN \leq k \leq bN} N_\beta \mathbb{E}^N \big(\eta^l_t(k)\big)^2 \to 0$$

as $N \uparrow \infty$.

Sketch of 1-block estimate

We give the main idea to estimate

$$V_{k,l}(\eta) = \chi_{\eta(k)>0} - E_{k,l,(2l+1)\eta^l(k)}[\chi_{\eta(k)>0}]$$

where $E_{k,l,j}$ is the conditional expectation given that there are j particle in the l-block around k.

–A separate argument will show that we can replace conditional expectation by $\frac{\eta^l(k)}{1+\eta^l(k)}$.

By an inequality implied by the definition of relative entropy,

$$\sup_{aN \le k \le bN} \mathbb{E}^N \left| \int_0^T N_\beta V_{k,l}(\eta_s) ds \right|$$

$$\le \frac{N_\beta H(\nu^N; \mathcal{R}_{c,N})}{\gamma N} + \sup_{aN \le k \le bN} \frac{N_\beta}{\gamma N} \ln \mathbb{E}_{\mathcal{R}_{c,N}} \exp \left\{ \gamma N \left| \int_0^T V_{k,l}(\eta_s) ds \right| \right\}.$$

We may compute that the relative entropy

$$H(\nu^N; \mathcal{R}_{c,N}) = O(NN_\beta^{-1}),$$

Also, the second term.may be bounded by Feynman-Kac formula in terms of

$$\frac{N_\beta}{\gamma N}\alpha_{N,l}$$

where

$$\alpha_{N,l} = \text{ largest eigenvalue of } N^2 L + \gamma N V_{k,l}.$$

Through some particle number truncations, we will need to estimate

$$N_\beta \sup_{j \le Cl} \sup_f \left\{ E_{k,l,j}[V_{k,l} f] - \frac{N^2}{\gamma N} D_{k,l,j}(\sqrt{f}) \right\}.$$

–Since $N_\beta = o(N)$, the Dirichlet form penalizes densities f which are not constant. Since $V_{k,l,j}$ is mean-zero with respect to $E_{k,l,j}$, more or less the estimate follows.

To make this more precise, we show that the spectral gap for the localized dynamics, where $aN \leq k \leq bN$, can be uniformly bounded below, over N, in terms of l.

Also, we note, with respect to the canonical measure with $j \leq Cl$ particles, $\|V_{k,l}\|_\infty = O(l)$.

–One can estimate the quantity in brackets by

$$\begin{aligned}
&\frac{\gamma N^{-1}}{1 - 2\|V_{k,l}\|_\infty \mathrm{gap}_{l,k,j}^{-1}\gamma N^{-1}} E_{k,l,j}[V_{k,l}(-L_{k,l}^{-1})V_{k,l}] \\
&\leq \frac{\gamma N^{-1}\mathrm{gap}_{l,k,j}^{-1}\|V_{k,l}\|_\infty^2}{1 - 2\|V_{k,l}\|_\infty \mathrm{gap}_{l,k,j}^{-1}\gamma N^{-1}} \\
&= O(N^{-1}),
\end{aligned}$$

enough to compensate for the N_β factor out front, a source of difficulty.

Some possible other directions

–Ensembles, as in Erlihson-Granovsky 2008, lead to other 'zero-range' interactions.

–Other limit theorems, fluctuations, large deviations, etc.

–We mention also that hydrodynamic limits for 3D (or 2+1) models related to Young diagrams have been shown recently, for instance

Borodin-Ferrari 2014,
Legras-Toninelli 2017,
Zhang 2018,
Laslier-Toninelli 2017.

–There are of course many interesting phenomena in the higher dimensional models to pursue.

Many of the references can be found in the arXiv version

I. Fatkullin–SS–J. Xue, 2018, arXiv:1809.03592

and also the surveys

–T. Funaki: *Lectures on Random Interfaces* Springer Briefs in Probability and Mathematical Statistics, 2016, Springer, Singapore

–F. Toninelli: (2+1)-dimensional interface dynamics: Mixing time, hydrodynamic limit and anisotropic KPZ growth, 2017, ICM article, arXiv:1711.05571v1