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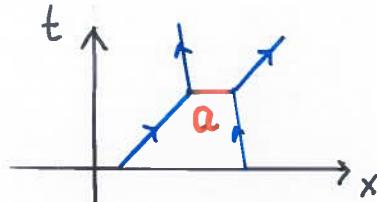
# Generalized Hydrodynamics and the Classical Toda Chain

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## integrable many-body

- hard rods



## generalized hydrodynamics

Percus 1969, HS 1982, Dobrushin, Boldrighini, Suhov 1983  
+ 1997

- KdV  $\partial_t u + 6u\partial_x u + \partial_x^3 u = 0$  Zakharov 1971, El et al. 2003 - now

- Lieb-Liniger

- $H_{xxz}$

- Fermi-Hubbard

}

2016 - now

Bertini, Collura, Doxon, De Nardis, Fagotti  
Karasch, Moore, Castro-Alvaredo, Yoshimura,  
Ilievski, Vasseur, ...

## "GHD on an atom chip"

Bouchoule et al., Oct. 2018

4000 Rb atoms, magnetic trap

1. GHD, general structure

2. classical Toda

Generalized Gibbs Ensemble

# 1. Hydrodynamics

1D, integrable, short range

$$H_{xxz} = \sum_j \{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \}$$

- hamiltonian  $H = \sum_j H_j$  local density
- conserved fields  $Q^{(n)} = \sum_j Q_j^{(n)}$   $n = 1, 2, \dots$  label?  
charges local COMPLETE list

- currents

$$\begin{aligned} \frac{d}{dt} Q_j^{(n)} &= i [H, Q_j^{(n)}] \\ &= J_j^{(n)} - J_{j+1}^{(n)} \end{aligned}$$

local



- GGE  $\frac{1}{Z} e^{\left(\sum_{n=1}^{\infty} \mu_n Q^{(n)}\right)}$   $\langle \cdot \rangle_{\tilde{\mu}}$  chemical potentials  $\mu_n$

- local GGE  $\frac{1}{Z} e^{\left(\sum_{n=1}^{\infty} \sum_j \mu_n (\epsilon_j) Q_j^{(n)}\right)}$   $\epsilon \ll 1$  slow variation

- hydrodynamics

$$\partial_t \langle Q_j^{(n)}(t) \rangle + \partial_j \langle J_j^{(n)}(t) \rangle = 0$$

local GGE

$$\varepsilon j = x, t \rightsquigarrow \varepsilon^{-1} t \quad \Downarrow \text{approximate}$$

$$\partial_t \langle Q_j^{(n)} \rangle_{\vec{\mu}(x,t)} + \partial_x \langle J_j^{(n)} \rangle_{\vec{\mu}(x,t)} = 0$$

INPUT       $\langle Q_0^{(n)} \rangle_{\vec{\mu}}$  ,  $\langle J_0^{(n)} \rangle_{\vec{\mu}}$       GGE

$$\langle Q_0^{(n)} \rangle_{\vec{\mu}} = \vec{u}, \vec{\mu} \Leftrightarrow \vec{v}, \vec{\mu} \quad , \quad \langle J_0^{(n)} \rangle_{\vec{\mu}} = \vec{j}(\vec{u})$$

$\Rightarrow$  coupled set of  
conservation laws

$$\partial_t \vec{u} + \partial_x \vec{j}(\vec{u}) = 0$$

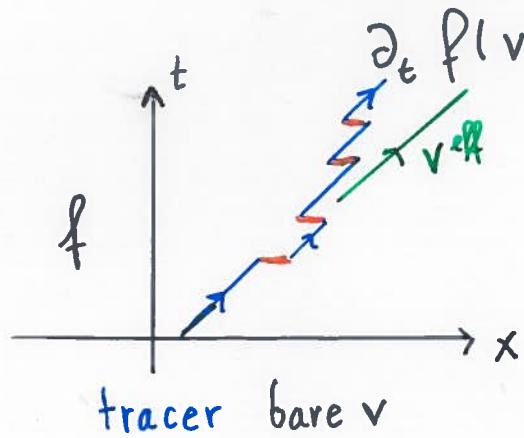
difficulty:  $\|\vec{j}(\vec{u})\|$

Example: hard rods      conserved fields      density  $\rho$ , velocity  $v$

$$f(x, t; v)$$

label  $\rightarrow$

$$\rho(x, t) = \int dv f(x, t; v), \quad u = \frac{1}{\rho} \int dv f(v)$$



$$\partial_t f(v) + \partial_x J_{[f]}(v) = 0$$

$$J_{[f]}(v) = v_{[f]}^{\text{eff}} f(v)$$

$v_{[f]}^{\text{eff}}$   
 $K(v)$

$$K(v) = v - \underbrace{\int dw a f(w) (\omega(w) - \omega(v))}_{\substack{\text{phase shift} \\ \text{collision rate}}}$$

$\Rightarrow$  in general:  $a \rightsquigarrow \phi(v, w)$

two-body phase shift

all quantum models

$\rightsquigarrow \left\| \partial_t f(v) + \partial_x \left( \frac{1}{1-\alpha\rho} (v - \alpha\rho u) f(v) \right) = 0 \right\|$

"educated guess"

## 2. Toda chain

$$H_N = \sum_{j=1}^N \left( \frac{1}{2} p_j^2 + e^{-r_j} \right)$$

stretch  $r_j = q_{j+1} - q_j$

Henon, Flaschka 1974

periodic b.c.

$$a_j = \frac{1}{2} e^{-r_j/2}, \quad b_j = \frac{1}{2} p_j$$

Lax Matrix  $N \times N$

$L_N$  tridiagonal,  $L_N = (L_N)^T$

$$\begin{bmatrix} & & & a_j & & 0 \\ & & & b_j & & \\ & & & a_j & & \\ & & & & & \\ & & & & & \\ 0 & & & & & \end{bmatrix}$$

$B_N$  matrix  $(B_N)^T = -B_N$

$$\begin{bmatrix} & & & -a_j & & 0 \\ & & & 0 & & \\ & & & a_j & & \\ & & & & & \\ & & & & & \\ 0 & & & & & \end{bmatrix}$$

$$\frac{d}{dt} L_N = [B_N, L_N]$$

$N \rightarrow \infty$   $L, B$  two-sided Jacobi matrices

- conserved fields

$$Q^{(n)} = \text{tr}[(L_N)^n] \quad \parallel \text{density} \quad Q_j^{(n)} = (L_N)_{jj}^n$$

$$n = 1, 2, \dots$$

range  $[j-n, j+n]$

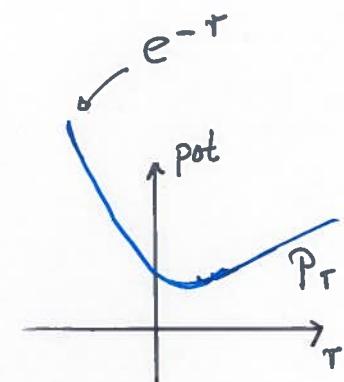
- currents  $J_j^{(n)} = (L^n L^{\text{off}})_{jj}$  local

IN ADDITION stretch  $\tau_j$ , dual pressure  $P$

$$P > 0$$

$$\text{current } p_j = 2(Q^{(n)})_{jj}$$

$$e^{-P\tau_j}$$



- GGE partition function

$$Z_N = \int \prod_{j=1}^N dp_j dq_j e^{-P\tau_j} e^{-\text{tr}[V(L_N)]}$$

$$\sum_{n=1}^N \mu_n Q_N^{(n)}$$

$$\left( V(x) \geq c_0 + c_1 |x| \right)$$

$$c_1 > 0$$

free energy:  $F_{[V]}(P) = \lim_{N \rightarrow \infty} -\frac{1}{N} \log Z_N$

RESULT

V fixed

$$\mathcal{F}^{\text{MF}}(\rho) = \int dx V(x) \rho(x) - \int dx \int dy \log|x-y| \rho(x) \rho(y) + \int dx \rho(x) \log \rho(x)$$

$$\text{minimize } \rho \geq 0, \int dx \rho(x) = P \Rightarrow \rho^*(x, P)$$

Lagrange multiplier  $\lambda$  minimizer  $\rho_\lambda^*(x)$

Euler-Lagrange

$$V(x) - 2 \int dy \log|x-y| \rho_\lambda^*(y) + \log \rho_\lambda^*(x) - \lambda + 1 = 0$$

classical limit TBA

NO Yang-Yang (dressing)

$$P = \int dx \rho_{\lambda(P)}^*(x)$$

$$\Rightarrow \left\| \mathcal{F}_{\text{toda}}(P) = \lambda(P) - (1+2P) \log 2 - 1 \right\|$$

$$L^n : \langle Q_0^{(n)} \rangle = \int dx \partial_p g^*(x, P) x^n \quad \leftrightarrow \text{GGE average } P, V$$

$\partial_p g^*(x, P)$  is the DOS of  $L$  under GGE with  $P, V$

slowly varying in space-time

stretch:  $\langle r_0 \rangle = \chi'(P) - 2 \log 2$

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exact solution  $V(x) = \frac{1}{2} x^2$

Opper 1985, see book Toda 1989

Toda chain

Allez, Bouchaud, Guionnet 2012

RMT

$$\frac{1}{P} g^*(x, P) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} |\hat{f}_P(x)|^{-2}$$

normalized

$$\hat{f}_P(x) = \int_0^\infty dt \left(\frac{P}{T(P)}\right)^{1/2} t^{P-1} e^{-\frac{1}{2}t^2} e^{ixt}$$

$P \rightarrow 0$  Gauss ,  $P \rightarrow \infty$  Wigner semi-circle

method

$$V(x) = \frac{1}{2}x^2 + \tilde{V}(x)$$

$$L_N \text{ under } \frac{1}{Z} e^{-\frac{1}{2}\text{tr}[(L_N)^2]}$$

$$L_{jj} = N(0,1) \left( \frac{1}{\sqrt{2\beta}} e^{-x^2/2} \right) \text{ i.i.d.}$$

$$L_{j,j+1} = \frac{1}{\sqrt{2}} \times_{2\beta} \left( \frac{1}{2 \cdot 4^{\beta} T(\beta)} x^{2\beta-1} e^{-x^2}, x > 0 \right) \text{ i.i.d.}$$

$$Z_N = \langle e^{-\text{tr}[\tilde{V}(L_N)]} \rangle_{(\beta)}$$

Dumitriu, Edelman 2002

$$T_{N,\beta} : (T_{N,\beta})_{jj} = N(0,1) \text{ i.i.d.}$$

$$T = (T)^T$$

$$(T_{N,\beta})_{j,j+1} = \frac{1}{\sqrt{2}} \times_{(N-j)\beta} \underbrace{\text{linear ramp}}_{\text{ramp}} \text{ i.d.}$$

≈ eigenvalues  $\lambda_1, \dots, \lambda_N$

distributed according to

$\beta$ -ensemble

$$\frac{1}{Z} \prod_{j=1}^N e^{-\frac{1}{2}\lambda_j^2} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta$$

SET

$$\beta = \frac{2\beta}{N}$$



$$\lim_{N \rightarrow \infty} -\frac{1}{N} \log \langle e^{-\text{tr}[\tilde{V}(T_N)]} \rangle(P) = \int_0^1 d\alpha \lim_{N \rightarrow \infty} -\frac{1}{N} \log \langle e^{-\text{tr}[\tilde{V}(L_N)]} \rangle(\alpha P)$$

||

linear ramp

$f_{[V]}^{MF}(P)$

→ rule  $\partial_P P$

$$Z_N = \int d\lambda_1 \dots d\lambda_N e^{-\left( \sum_{j=1}^N V(\lambda_j) - \frac{P}{N} \sum_{i \neq j=1}^N \log |\lambda_i - \lambda_j| \right)}$$

- mean, covariance of conserved fields

### 3. MF Dyson Brownian motion

$\beta$ -ensemble Israelsson 2001  
works also for MF

$$dx_j(t) = -\frac{1}{2} V'(x_j) dt + \frac{\mathbb{P}}{N} \sum_{\substack{i=1 \\ i \neq j}}^N \frac{1}{x_i(t) - x_j(t)} dt + dB_j(t)$$

empirical measure  $\frac{1}{N} \sum_{j=1}^N f(x_j(t)) = \int \rho^N(dx, t) f(x)$

- $\rho^N \rightarrow \rho$  a.s.

$$\partial_t \rho = \partial_x \left( \frac{1}{2} V' - \mathbb{P} \int dy \frac{1}{x-y} \rho(y, t) \rho + \frac{1}{2} \partial_x \rho \right)$$

- stationary measure  $\frac{1}{Z} \prod_{j=1}^N e^{-V(x_j)} e^{\frac{\mathbb{P}}{N} \sum_{i \neq j=1}^N \log |x_i - x_j|} = \mu_{N \text{ stat}}$

$$\lim_{N \rightarrow \infty} \mu_{N \text{ stat}} \restriction_{[1, \dots, m]} = (\rho_{\min})^{\otimes m}$$

- stationary process fluctuations  $\frac{1}{\sqrt{N}} \sum_{j=1}^N f(x_j | t) - \int dx p_{\min}(x) f(x)$

CLT

limit OU-process

$$\frac{d}{dt} \phi(\rho, t) = \phi \left( \frac{1}{2} (\partial_x - V'(x)) \partial_x \rho, t \right) + \frac{1}{2} \mathbb{P} \int dx \int dy \underbrace{\frac{\partial_x \rho(x) - \partial_y \rho(y)}{x-y} \left( p_{\min}(x) \phi(y, t) + \phi(x, t) p_{\min}(y) \right)}_{\text{Linearized}} + \tilde{\zeta} \left( \sqrt{p_{\min}} \partial_x \rho, t \right)$$

$\tilde{\zeta}$  space-time white noise

## Outlook

- GGE - average currents ??
- linearized hydrodynamics (only  $P$ ,  $\frac{1}{T}$  thermal) with C. Mendl  
MD simulations Dhar et al. 2016  
 $\Rightarrow$  to probe validity of hydrodynamic approach ←
- comparison with GHD of quantum models