A Quadratic Size-Hierarchy Theorem for Small-Depth Multilinear Formulas

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Joint work with

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Fundamental Question

$$\textbf{More resources} \overset{?}{\Longrightarrow} \textbf{More computational power.}$$

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Answer: Yes in many cases.
Eg., *Time Hierarchy* and *Space Hierarchy* theorems in the classical complexity.

Classical Hierarchy Theorems over Turing Machines

Time Hierarchy Theorem

For every $t(n)$ and $\delta > 0$, there is a decision problem which can be solved in time $t(n)$ but not in the time $t(n)^{1-\delta}$, i.e., $\mathsf{DTIME}(t(n)^{1-\delta}) \subsetneq \mathsf{DTIME}(t(n))$.

Space Hierarchy Theorem

For every $s(n)$ and $\delta > 0$, there is a language L that is decidable in space $s(n)$ but not in space $s(n)^{1-\delta}$, i.e., $\mathsf{SPACE}(s(n)^{1-\delta}) \subsetneq \mathsf{SPACE}(s(n))$.

Generalized Meta Theorem for Any Resource

For every $f(n)$, there is a function that can be computed using $f(n)$ resources but cannot be computed using $\ll f(n)$ resources.

This gives us a strict computational hierarchy between $\ll f(n)$ resources and $f(n)$ resources.

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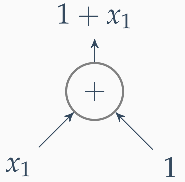
- **Our goal:** Similar theorems for *Arithmetic Formulas*.
- **Resource:** *Size* of the arithmetic formula, which corresponds to the maximum number of arithmetic operations.

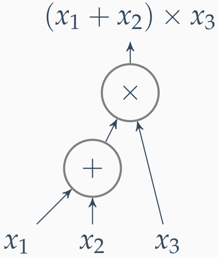
Computing polynomials syntactically

Definition

An Arithmetic Formula Φ over the field $\mathbb{F}$ and the set of variables $X = (x_1, x_2, \ldots, x_n)$ is a *directed tree* as follows:

- Leaf nodes are labelled either by a variable or a field element from $\mathbb{F}$ and the root node outputs the polynomial.
- Every other node is labelled by either $\times$ or $+$).
- The size of Φ is the number of nodes present in it.
- The depth of Φ is the length of the longest leaf to root path.





Size Hierarchy for Arithmetic Formulas

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Fundamental Question Rephrased: Size Hierarchy

For any $\delta > 0$ and $s = n^c$, show that there is a polynomial P_n that it is computed by a formula of size $s(n)$ but not by formulas of size $s(n)^{1-\delta}$.

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- No techniques are available to prove size lower bounds any better than $\tilde{\Omega}(n^3)$ [Kayal et al., 2016, Balaji et al., 2016] for small depth circuits and $\Omega(n^2)$ [Kalorkoti, 1985] for general formulas.
- Some techniques are available to prove lower bounds against formulas when every computation is restricted to be *multilinear*.

Multilinear polynomial

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Syntactic Multilinearity

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Syntactic Multilinearity

A product is said to be syntactically multilinear if the inputs are defined over disjoint sets of variables.

$$(x_1 + x_2)(x_1 + x_3) - (x_1 + x_4)(x_1 + x_2) = x_1x_3 + x_2x_3 - x_1x_4 - x_2x_4.$$

This is not a syntactically multilinear computation.

Size Hierarchy for Multilinear Formulas

Theorem ([Raz, 2004, Raz and Yehudayoff, 2008])

For any $s = n^c$ where c is a fixed constant, there is an explicit polynomial that can be computed by a multilinear arithmetic formula of size $s(n)$ but not by any multilinear arithmetic formulas of size $s(n)^\alpha$ where $\alpha \leq 1/30$.

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Theorem (This work)

For any $\delta \in (0, 1/2)$ and $s(n) = n^c$ for some fixed constant c, there is an explicit polynomial that can be computed by a multilinear arithmetic formula of size $s(n)$ and depth-3 but not by any multilinear formulas of size $s^{0.5-\delta}$ and depth $O(\log s / \log \log s)$.

Related Work

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[Raz, 2004, Raz and Yehudayoff, 2008]	This Work
There is no restriction on the depth of multilinear formulas.	👎 We can prove a size lower bound only when the depth is $O(\log s / \log\log s)$.
The separation they show is s vs s^{α} where $\alpha < 1/30$, even at small-depths.	👍 At small-depths, we show a better separation of s vs $s^{1/2-\delta}$.
The hard polynomial has a formula of size s and depth $\Omega(\sqrt{\log s})$.	👍 The hard polynomial has a formula of size s and depth 3.

Tools & Techniques

Theme of the proofs

- We can define a suitable complexity measure $\mu : \mathbb{F}[X] \mapsto \mathbb{N}$ such that the following holds:
  - If f is computed by a *small*-depth multilinear formula then $\mu(f)$ is *small*.
  - For the hard polynomial P, $\mu(P)$ is *large*.

Tool 1: Partial Derivative Matrix & Complexity Measure

Following Raz [Raz, 2004], we too use the rank arguments.

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$$f = \sum_{i=1}^{2^n} c_i \cdot m_i \quad \mapsto \quad f|_\rho = \sum_{i=1}^{2^n} c_i \cdot m_{i,Y} \cdot m_{i,Z}$$

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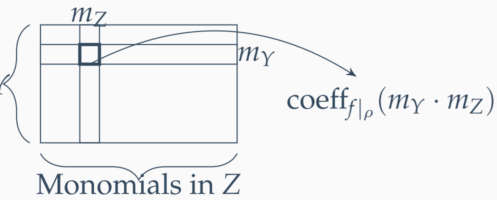
$M_{(Y,Z)}(f|_\rho)$: Monomials in Y



Complexity of f under ρ is $\text{rank}(M_{(Y,Z)}(f|_\rho))$.

Fact: $\text{rank}(M_{(Y,Z)}(f|_\rho)) \leq 2^{\frac{|Y|+|Z|}{2}}$.

Example

Consider the polynomial $f(x_1, x_2) = (x_1 + x_2)$ and the partition map of $\{x_1, x_2\}$ such that

$$x_1 \mapsto y; \quad x_2 \mapsto z.$$

It follows that $f|_\rho = (y + z)$ and thus,

$$M_{(\{y\},\{z\})}(f|_\rho) = \begin{array}{c} \\ 1 \\ y \end{array} \begin{array}{c} \begin{matrix} 1 & z \end{matrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array},$$

$$\text{rank}(M_{(\{y\},\{z\})}(f|_\rho)) = 2.$$

Example

Consider the polynomial $f(x_1, x_2, x_3, x_4) = (x_1 + x_2)(x_3 + x_4)$ and the partition map of $\{x_1, x_2, x_3, x_4\}$ such that

$$x_1 \mapsto y_1; \quad x_2 \mapsto z_1; \quad x_3 \mapsto y_2 \quad ; x_4 \mapsto z_2.$$

It follows that $f|_\rho = (y_1 + z_1)(y_2 + z_2) = y_1y_2 + y_1z_2 + z_1y_2 + z_1z_2$ and thus,

$$M_{(\{y_1,y_2\},\{z_1,z_2\})}(f|_\rho) = \begin{array}{c} \\ 1 \\ y_1 \\ y_2 \\ y_1y_2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & z_1 & z_2 & z_1z_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

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$$\operatorname{rank}(M_{(\{y_1,y_2\},\{z_1,z_2\})}(f|_\rho)) = 1.$$

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- Let the variable mapping under ρ be the following.

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\begin{array}{ccccccc}
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- Under ρ, it is easy to see that the polynomial defined as follows will have full rank.

$$\Gamma_\rho(X) = (x_{i_1} + x_{j_1})(x_{i_2} + x_{j_2}) \cdots (x_{i_m} + x_{j_m})$$
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- Γ_ρ has a very small formula.

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  3. Construct a polynomial from S as defined above.

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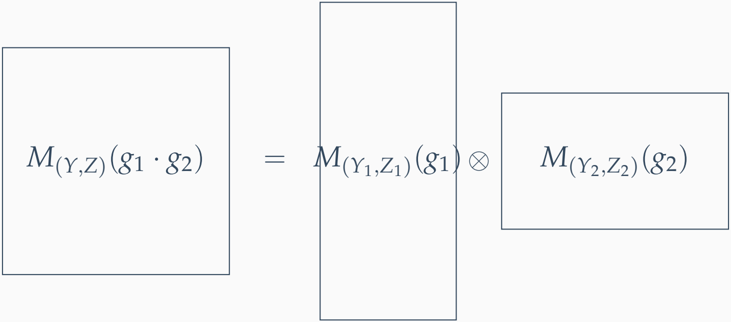

$$
\begin{aligned}
\operatorname{rank}(M_{(Y,Z)}(g_1 \cdot g_2)) &= \operatorname{rank}(M_{(Y_1,Z_1)}(g_1)) \cdot \operatorname{rank}(M_{(Y_2,Z_2)}(g_2)) \\
&\leq 2^{\frac{|Y_1|+|Z_1|-1}{2}} \cdot 2^{\frac{|Y_2|+|Z_2|-1}{2}} = 2^{\frac{|Y|+|Z|}{2}-1}.
\end{aligned}
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- If ρ is such that there there ℓ factors f_i such that $|Y_i| \neq |Z_i|$, we get that

$$\text{rank}(M_{(Y,Z)}(f|_\rho)) \leq 2^{\frac{|Y|+|Z|}{2} - \frac{\ell}{2}}.$$

Tool 2: Product decomposition of Multilinear Formulas

Lemma (Product Decomposition, [Shpilka and Yehudayoff, 2010])

Any multilinear formula of size s_0 and product depth Δ, over n variables can be decomposed into a sum of $s = s_0 n$ many products each of which has a lot of factors.

$$f = \sum_{i=1}^{s} f_i = \sum_{i=1}^{s} f_{i,1} \cdot f_{i,2} \cdot \ldots \cdot f_{i,t} \text{ where } t \geq n^{1/2\Delta}.$$

and

- *for all $i \in [s]$ and $j \in [t]$, $\left|\mathrm{vars}(f_{i,j})\right| > 1$,*
- *for all $i \in [s]$, $f_{i,1}, f_{i,2}, \cdots, f_{i,t}$ are defined over disjoint sets of variables.*

Subadditivity of rank

Let $g, g_1, g_2, \cdots, g_r$ be polynomials over $\mathbb{F}[Y \cup Z]$ such that

$$g = \sum_{i \in [r]} g_i$$

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$$\operatorname{rank}(M_{(Y,Z)}(g)) \leq \sum_{i \in [r]} \operatorname{rank}(M_{(Y,Z)}(g_i)).$$

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$$\operatorname{rank}(M_{(Y,Z)}(f|_\rho)) \leq \sum_{i \in [s]} \operatorname{rank}(M_{(Y,Z)}(f_i|_\rho)) \leq s \cdot 2^{\frac{|Y|+|Z|}{2} - \frac{\ell}{2}}.$$

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$$\text{rank}(M_{(Y,Z)}(f|_\rho)) \leq \sum_{i \in [s]} \text{rank}(M_{(Y,Z)}(f_i|_\rho)) \leq s \cdot 2^{\frac{|Y|+|Z|}{2} - \frac{\ell}{2}}.$$

We want $s \cdot 2^{\frac{|Y|+|Z|}{2} - \frac{\ell}{2}}$ to be strictly less than $2^{\frac{|Y|+|Z|}{2}}$ and thus we want $\ell > 2 \log s$.

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For a set of partitions $S = \{\rho_1, \rho_2, \cdots, \rho_m\}$:

- E_i is also defined by a system of linear equations.
- It is sufficient to show that, for each i,

$$\mathbb{P}_{\rho \in S}\left[E_i\right] < 1/s.$$

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Probabilistic Method	Our derandomization using subspace evading sets
$m = O(ns)$	$m = O(ns^2)$

Thank you![*][†]

[*] Figure of the coefficient matrix were sourced from Ramprasad Saptharishi's survey, under CC BY-NC-SA licence.

[†] The theme of these slides is based on mtheme by matze https://github.com/matze/mtheme, under CC BY-SA licence.